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1. Show that the Fourier transform of the distribution $|x|^m$ is

$$\begin{cases} (-i)^m 2\pi\delta^{(m)}(k), & m = 2,4,6,\dots, \text{ and} \\ -2(-i)^{m+1}\frac{d^m}{dk^m}\text{PV}(\frac{1}{k}), & m = 1,3,5,\dots \end{cases} \tag{1}$$

Solution. Start with the analog of the relationship

$$\widehat{\frac{d^m f(x)}{dx^m}} = (-ik)^m \hat{f}(k) \tag{2}$$

for distributions. We have for the distributional derivatives of $|x|$ and, subsequently, $|x|^m$:

$$\frac{d|x|}{dx} = \text{sgn}(x) \tag{3}$$

$$\frac{d|x|^m}{dx} = m|x|^{m-1}\text{sgn}(x). \tag{4}$$

Before beginning the next step, the following formulae are useful:

$$\text{sgn}'(x) = 2\delta(x) \tag{5}$$

$$\widehat{\text{sgn}}(k) = 2i\text{PV}(\frac{1}{k}). \tag{6}$$

Perform the derivative m times and apply relationship (2) to get the following:

$$\widehat{|x|^m} = \frac{2i(m!)\text{PV}(\frac{1}{k})}{(-ik)^m} \tag{7}$$

where all subsequent transform integrals except the above contained $\delta(x)$ terms, which when coupled with $f(x) = |x|e^{ikx}$ led to $\int$'s valued 0, since $f(0) = 0$.

Now, apply the definition of the PV and δ functions as the following sequence limits:

$$\text{PV}(\frac{1}{k}) = \lim_{\epsilon\to 0} \frac{k}{k^2+\epsilon^2} \tag{8}$$

$$\delta(k) = \lim_{\epsilon\to 0} \frac{1}{\pi}\frac{\epsilon}{k^2+\epsilon^2}. \tag{9}$$

Perform the derivative m times on both formulae, and the following emerges:

$$\widehat{|x|^m} = \begin{cases} (-i)^m 2\pi\delta^{(m)}(k), & m = 2,4,6,\dots, \text{ and} \\ -2(-i)^{m+1}\frac{d^m}{dk^m}\text{PV}(\frac{1}{k}), & m = 1,3,5,\dots \end{cases} \tag{10}$$

which is the desired result.

2. Distributions $T_t^{\pm}$ are defined by

$$(T_t^{\pm}, \psi) := \lim_{\epsilon \downarrow 0} \left(\int_{-\infty}^{-\epsilon} \frac{e^{\pm itx}\psi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{e^{\pm itx}\psi(x)}{x} dx \right). \tag{11}$$

In these formulas, t is a parameter. Show that

$$\lim_{t\to\infty} T_t^{+} = i\pi\delta, \; \lim_{t\to\infty} = -i\pi\delta \tag{12}$$

Solution. Start with T_t^{-}:

$$\lim_{t\to\infty} T_t^{-} = \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} \frac{e^{-itx}\psi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{e^{-itx}\psi(x)}{x} dx \right) \tag{13}$$

$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(\int_{\infty}^{\epsilon} \frac{e^{itx}\psi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{e^{-itx}\psi(x)}{x} dx \right) \tag{14}$$

$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(-\int_{\epsilon}^{\infty} \frac{e^{itx}\psi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{e^{-itx}\psi(x)}{x} dx \right) \tag{15}$$

$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(\int_{\epsilon}^{\infty} \frac{(e^{-itx} - e^{itx})\psi(x)}{x} dx \right) \tag{16}$$

$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(-2i \int_{\epsilon}^{\infty} \frac{\sin(tx)\psi(x)}{x} dx \right) \tag{17}$$

$$\tag{18}$$

but, this is (using a relation from Tabor (5.58)):

$$= -2i\frac{\pi}{2}\text{sgn}(t)\delta = -i\pi\delta \tag{19}$$

The same relation is used for T_t^{+}, with a difference in sign:

$$\lim_{t\to\infty} T_t^{-} = \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} \frac{e^{itx}\psi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{e^{itx}\psi(x)}{x} dx \right) \tag{20}$$

$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(\int_{\infty}^{\epsilon} \frac{e^{-itx}\psi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{e^{-itx}\psi(x)}{x} dx \right) \tag{21}$$

$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(-\int_{\epsilon}^{\infty} \frac{e^{-itx}\psi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{e^{itx}\psi(x)}{x} dx \right) \tag{22}$$

$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(\int_{\epsilon}^{\infty} \frac{(e^{itx} - e^{-itx})\psi(x)}{x} dx \right) \tag{23}$$

$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\left(2i \int_{\epsilon}^{\infty} \frac{\sin(tx)\psi(x)}{x} dx \right) \tag{24}$$

$$= 2i\frac{\pi}{2}\text{sgn}(t)\delta = i\pi\delta \tag{25}$$

3. Object: to represent solutions of the differential equation

$$y'''(x) - xy(x) = 0 \tag{26}$$

by contour integrals

$$y(x) = \int_{C_j} f(s)e^{sx} ds. \tag{27}$$

Here C_j is a contour, to be determined, that extends to ∞ in both directions.
a.) Find $f(s)$.

Solution. Plug (27) into the ODE (26):

$$\int_{C_j} (s^3 - x) f(s) e^{sx} ds = 0. \tag{28}$$

Look at the integral with the x term in front:

$$\int_{C_j} x f(s) e^{sx} ds = f(s)e^{sx}\Big|_{u_1}^{u_2} - \int_{C_j} f'(s)e^{sx} ds \tag{29}$$

where u_1, u_2 are the endpoints of the yet-to-be determined integration contour. Combine (29) with remaining terms of (28):

$$-f(s)e^{sx}\Big|_{u_1}^{u_2} + \int_{C_j} (s^3 f(s) + f'(s))e^{sx} ds \quad = \quad 0 \tag{30}$$

$$(\implies) \quad f'(s) + s^3 f(s) \quad = \quad 0 \tag{31}$$

$$(\implies) \quad f(s) = ae^{-\frac{s^4}{4}} \quad = \quad e^{-\frac{s^4}{4}}. \tag{32}$$

Assume WLOG that $a = 1$.

b.) Find contours C_j that can be used to get solutions represented in the form (27).

Solution. Need to find contours with boundaries at u_1, u_2 such that $f(s)e^{sx}\Big|_{u_1}^{u_2} = 0$. First, recast $s = Re^{i\theta}$ and plug into $f(s)e^{sx} = e^{sx - \frac{s^4}{4}}$.

$$sx - \frac{s^4}{4} = Rx(\cos\theta + i\sin\theta) - \frac{R^4}{4}(\cos 4\theta + i \sin 4\theta) \tag{33}$$

Take the modulus, and look at the exponent

$$e^{\left|sx - \frac{s^4}{4}\right|} \le e^{R|x|\cos\theta - \frac{R^4}{4}\cos 4\theta} \tag{34}$$

We need $\cos 4\theta > 0$ to ensure that that the exponential decays to 0 as $R \to \infty$. There are four regions:

$$
\begin{aligned}
-\frac{\pi}{2} &< 4\theta < \frac{\pi}{2}, \quad \theta \in \left(-\frac{\pi}{8}, \frac{\pi}{8}\right) && (35)\\
\frac{3\pi}{2} &< 4\theta < \frac{5\pi}{2}, \quad \theta \in \left(\frac{3\pi}{8}, \frac{5\pi}{8}\right) && (36)\\
\frac{7\pi}{2} &< 4\theta < \frac{9\pi}{2}, \quad \theta \in \left(\frac{7\pi}{8}, \frac{9\pi}{8}\right) && (37)\\
\frac{11\pi}{2} &< 4\theta < \frac{13\pi}{2}, \quad \theta \in \left(\frac{11\pi}{8}, \frac{13\pi}{8}\right) && (38)\\
& && (39)
\end{aligned}
$$

The points u_1, u_2 are the endpoints of the contours at ∞.

4. Let A be the operator $(Au)(x) = -x(xu'(x))'$ on $L^2([0,\infty))$, with boundary condition $u(0) = 0$.
a.) Find Green's function $G(x,\xi;\lambda)$ (the solution of $\lambda u(x) - (Au)(x) = \delta(x-\xi)$). This is rather messy. Be careful with branches of $\sqrt{\lambda}$.

Solution. To find Green's function satisfying the boundary condition $u(0) = 0$, take the following from Tabor (7.97,7.98):

$$G(x,\xi) = \begin{cases} 0 & x < \xi \\ \frac{u_1(x)u_2(\xi) - u_2(x)u_1(\xi)}{p(\xi)W(\xi)} & x > \xi. \end{cases} \tag{40}$$

Need to find the eigenfunctions $u_1(x;\lambda), u_2(x;\lambda)$ and the Wronskian at ξ, $W(\xi)$; $p(\xi) = \xi^2$.

$$\begin{aligned} x^r[r(r-1) + r + \lambda] &= 0 && (41) \\ x^r[r^2 + \lambda] &= 0 && (42) \\ r &= \pm i\lambda && (43) \\ (\implies)\ \ u_1(x;\lambda) = x^{i\sqrt{\lambda}},\ u_2(x;\lambda) &= x^{-i\sqrt{\lambda}} && (44) \\ W(\xi) = u_1(\xi)u_2'(\xi) - u_2(\xi)u_1'(\xi) &= -2i\sqrt{\lambda}\xi^{-1}. && (45) \end{aligned}$$

Combine with (37) to get:

$$G(x,\xi) = \begin{cases} 0 & x < \xi \\ \frac{x^{-i\sqrt{\lambda}}\xi^{i\sqrt{\lambda}} - x^{i\sqrt{\lambda}}\xi^{-i\sqrt{\lambda}}}{2i\xi\sqrt{\lambda}} & x > \xi. \end{cases} \tag{46}$$

b.) Do the appropriate integral of $G(x,\xi;\lambda)$ around a contour to get the following representation of $\delta(x-\xi)$:

$$\delta(x-\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \xi^{s-1}x^{-s}ds. \tag{47}$$

You will have to do a lot of simplifications of integrals. The final step is a change of variables from real to imaginary; this leads to the integration along the imaginary axis.

Solution. Make the substitution $k = \sqrt{\lambda}$ and the integral $\int_C G(x,\xi;\lambda)d\lambda$ becomes:

$$\delta(x-\xi) = \frac{\xi^{-1}}{2\pi i}\int_C \frac{x^{-i\sqrt{\lambda}}\xi^{i\sqrt{\lambda}} - x^{i\sqrt{\lambda}}\xi^{-i\sqrt{\lambda}}}{2i\sqrt{\lambda}}d\lambda = \frac{\xi^{-1}}{2\pi i}\int_C \frac{x^{-ik}\xi^{ik} - x^{ik}\xi^{-ik}}{i}dk \tag{48}$$

where the integral is taken over the whole real line. To ensure integrability split the integral and contour into:

$$\begin{aligned} &= \frac{\xi^{-1}}{2\pi i}\int_0^{\infty} \frac{x^{-ik}\xi^{ik}}{i}dk - \int_0^{-\infty} \frac{x^{ik}\xi^{-ik}}{i}dk && (49) \\ &= \frac{\xi^{-1}}{2\pi i}\int_0^{\infty} \frac{x^{-ik}\xi^{ik}}{i}dk + \int_{-\infty}^{0} \frac{x^{ik}\xi^{-ik}}{i}dk. && (50) \end{aligned}$$

Make substitution $s = -ik$. Now the integral is across the whole complex line:

$$\delta(x-\xi) = \frac{\xi^{-1}}{2\pi i}\int_{-i\infty}^{\infty} x^{-s}\xi^{s}ds = \frac{1}{2\pi i}\int_{-i\infty}^{\infty} x^{-s}\xi^{s-1}ds \tag{51}$$

c.) Derive the transform formulas

$$F(s) = \int_0^\infty x^{s-1}f(x)dx,\ f(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} x^{-s}F(s)ds. \tag{52}$$

F is called the *Mellin transform* of f.

Solution. Use above relation (48) and multiply by $f(\xi)$ and integrate over $L^2([0,\infty))$.

$$f(x) = \int_0^\infty f(\xi)\delta(x-\xi)d\xi = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} x^{-s}\left[\int_0^\infty f(\xi)\xi^{s-1}d\xi\right]ds. \tag{53}$$

Define:

$$F(s) = \int_0^\infty f(x)x^{s-1}dx, \text{ then} \tag{54}$$

$$f(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} x^{-s}F(s)ds. \tag{55}$$

d.) Find the Mellin transform of e^{-x}.

Solution. The integral for $F(s)$ is:

$$\int_0^\infty x^{s-1}e^{-x}dx. \tag{56}$$

This integral looks familiar, in fact it is the gamma function, therefore $F(s) = \Gamma(s)$.

If we didn't have the definition of the gamma function, we could use integration-by-parts and the fact that e^{-x} decays faster than powers of x. The result would be similar to the factorial function (and for the case of $s \in \mathbb{N}^+$ it would be the factorial function).

5. Let K be the integral operator

$$(Ku)(x) = \int_0^\infty e^{-x} e^{-2\xi} u(\xi) d\xi \tag{57}$$

on $L^2([0, \infty))$.

a.) For what λ does the equation $u(x) = \lambda(Ku)(x)$ have a nontrivial solution?

Solution. Multiply both sides by e^{-2x} and integrate. Look at $\int_0^\infty e^{-2x} u(x) dx$:

$$\int_0^\infty e^{-2x} u(x) dx = \lambda \int_0^\infty e^{-3x} dx \int_0^\infty e^{-2\xi} u(\xi) d\xi \tag{58}$$

$$= \frac{\lambda}{3} \int_0^\infty e^{-2\xi} u(\xi) d\xi. \tag{59}$$

This reduces to the simple scalar equation:

$$a = \frac{\lambda}{3} a \tag{60}$$

which has nontrivial solution only when $\lambda = 3$.

b.) Under what conditions on f will the equation

$$u(x) = \lambda(Ku)(x) + f(x) \tag{61}$$

have a solution, and is that solution unique? Consider all λ.

Solution. Follow the same steps as above:

$$\int_0^\infty e^{-2x} u(x) dx = \lambda \int_0^\infty e^{-3x} dx \int_0^\infty e^{-2\xi} u(\xi) d\xi + \int_0^\infty f(x) e^{-2x} dx \tag{62}$$

which leads to the algebraic equation:

$$a = \frac{\lambda}{3} a + c \tag{63}$$

$$(\implies) \; a = \frac{c}{1 - \frac{\lambda}{3}}. \tag{64}$$

We should require that f be an $L^1([0, \infty))$; that is, $c \neq \infty$. If $\lambda \neq 3$ and $c \neq 0$, a unique solution exists. If $\lambda = 3$ and $c \neq 0$, no solution exists. If $\lambda = 3$ and $c = 0$, infinitely many solutions exist.

c.) For each integer $n \geq 0$ find the solution, if one exists, of

$$u(x) = \lambda(Ku)(x) + e^{-nx}. \tag{65}$$

Consider all λ.

Solution. Use above equation (61) with $f(x) = e^{-nx}$:

$$\begin{aligned} a &= \frac{c}{1 - \frac{\lambda}{3}} &(66)\\ c &= \int_0^\infty e^{-(n+2)x} dx = \frac{1}{n+2}. &(67) \end{aligned}$$

Here $c \neq 0 \forall n$. Therefore a unique solution exists when $\lambda \neq 3$. The solution is given in Tabor (taken with 8.15) as:

$$u(x) = ae^{-x} + f(x) = \frac{c\lambda}{1 - \frac{\lambda}{3}} e^{-x} + e^{-nx} = \frac{\lambda}{(n+2)(1 - \frac{\lambda}{3})} e^{-x} + e^{-nx} \tag{68}$$

otherwise, no solution exists.

d.) In the case(s) where a solution of c.) does not exist, find a nonzero f for which there *is* a solution.

Solution. We need to pick a function that is orthogonal on $L^2([0, \infty))$ to e^{-2x}. Set up $f(x) = Ce^{-x} - 1$ and solve for C:

$$\begin{aligned} 0 &= \int_0^\infty (Ce^{-x} - 1)e^{-2x} dx = \frac{C}{3} - \frac{1}{2} &(69)\\ (\implies)\ C &= \frac{3}{2}. &(70) \end{aligned}$$

Therefore, if $f(x) = \frac{3}{2}e^{-x} - 1$, infinitely many solutions exist.

6. The Bessel function of the first kind of order n (a non-negative integer) has the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(nt - x\sin t)dt. \tag{71}$$

Find the leading asymptotic behavior as $x \to \infty$. (The answer should not have any $\sqrt{-1}$ in it.)

Solution. Take homework result

$$\begin{aligned} I(\tau) &\sim \sqrt{\frac{2\pi}{\tau S''(a)}} f(a) e^{i\tau S(a)} \int_0^{\infty} e^{isgn(S''(a))v'^2}\, dv' && (72) \\ &= \frac{1}{2}\sqrt{\frac{2\pi}{\tau S''(a)}} f(a) e^{i\tau S(a)} e^{i\frac{\pi}{4}sgn(S''(a))} && (73) \end{aligned}$$

for arbitrary a, with $S(t) = -\sin t$, $a = \frac{\pi}{2}$.

$$\begin{aligned} J_n(x) &= \frac{1}{2}\frac{1}{\pi}\mathbb{R}\left[\int_0^{\pi} e^{i(nt - x\sin t)} dt\right] && (74) \\ &\sim \frac{1}{2\pi}\mathbb{R}\left[\frac{1}{2}\sqrt{\frac{2\pi}{x\sin\frac{\pi}{2}}} e^{\frac{in\pi}{2}} e^{-ix\sin\frac{\pi}{2}} e^{\frac{i\pi}{4}}\right] && (75) \\ &= \frac{1}{2\pi}\mathbb{R}\left[\frac{1}{2}\sqrt{\frac{2\pi}{x}} e^{i\left(\frac{n\pi}{2} + \frac{\pi}{4} - x\right)}\right] && (76) \\ &= \frac{1}{2\pi}\mathbb{R}\left[\frac{1}{2}\sqrt{\frac{2\pi}{x}} e^{i\left(\frac{\pi}{4}\left(2n+1\right) - x\right)}\right] && (77) \\ &= \frac{1}{2}\sqrt{\frac{1}{2\pi x}} \cos\left(\frac{\pi}{4}\left(2n+1\right) - x\right) && (78) \end{aligned}$$

Therefore, leading asymptotic behavior as $x \to \infty$ is $\sim \frac{1}{\sqrt{x}}$